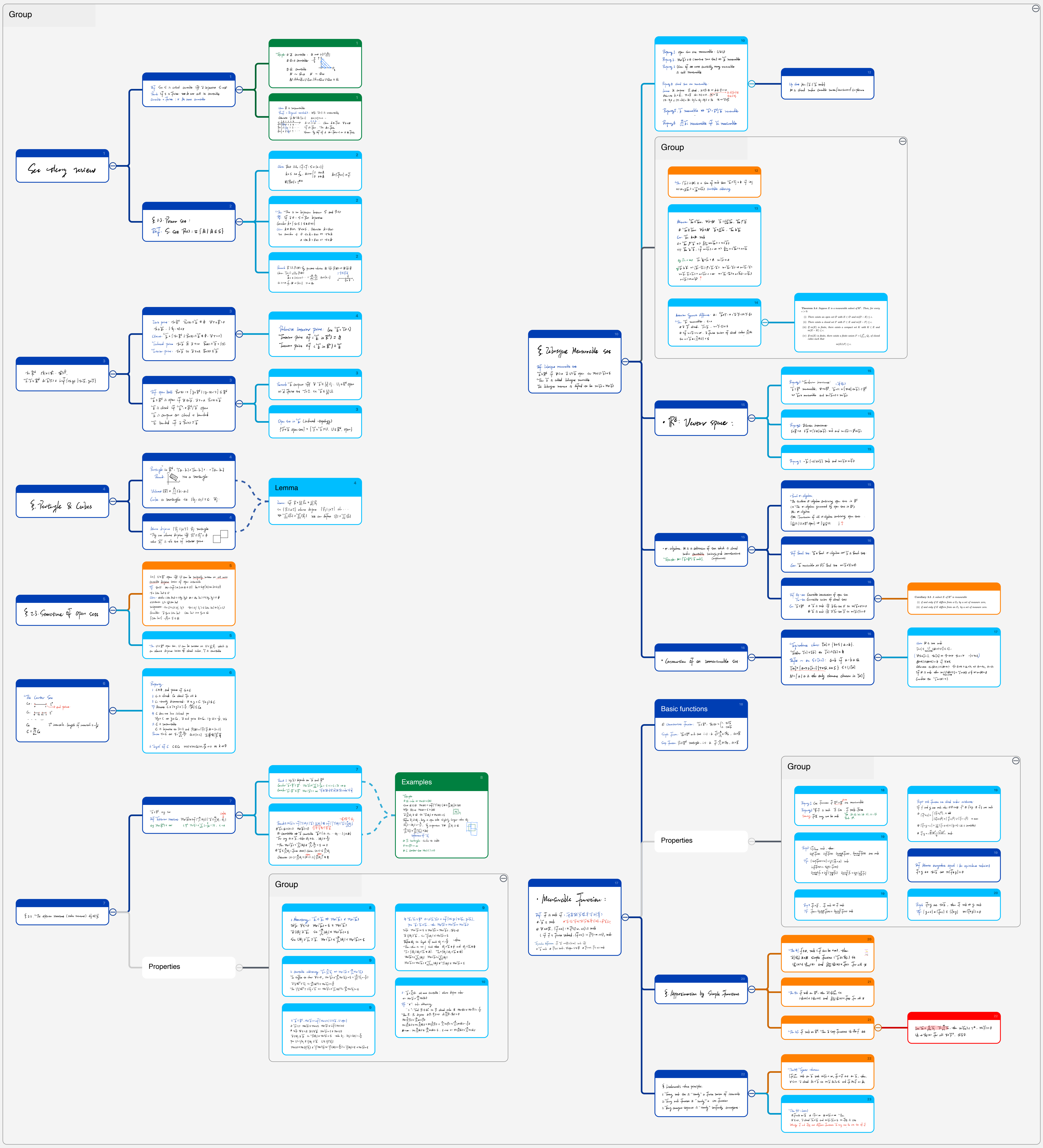

Group
Lemma
Examples
Group
Properties
Group
Basic functions
Properties
Group

Integration theory

Define integration

§. Integration Theory.

Overview: Integration of:

① Simple function ② msb f with $|f|<M$, $m(\mathrm{Supp} f)<\infty$

③ $f \geq 0$ ④ f msb.

Stage 1. Simple Functions.

Stage 2. bounded function supported on a set of finite measure

Goal: $\int f = \lim_{n\to\infty} \int \varphi_n$

Stage 3. non-negative functions

Def. $f \geq 0$. msb. then $\int f := \sup_g \int g$

where $0 \leq g \leq f$, $|g| < M$, $m(\mathrm{Supp}\, g) < \infty$

and $\int_E f = \int f \cdot \chi_E$. f is Lebesgue integrable if $\int f < \infty$

Stage 4. Any msb functions.

Def. f is integrable if $|f|$ is integrable

write $f^+ = \max\{0, f\}$. $f^- = \max\{0, -f\}$.

$f = f^+ - f^-$. $|f| = f^+ + f^-$ if $\int |f| < \infty \Rightarrow \int f^+, \int f^- < \infty$

Def. $\int f := \int f^+ - \int f^-$

f integrable $\Leftrightarrow$ f^+, f^- integrable.

§. L^1-Space

Def. $L^1(\mathbb{R}^d) = \{f \mid f \text{ integrable}\}/\sim = \{[f] \mid [f] = \{g \mid f = g \text{ a.e.}\}\}$

$f \sim g$: $f = g$ a.e.

Notation. $f \in L^1(\mathbb{R}^d)$: $[f] \in L^1(\mathbb{R}^d)$

· Generalization: $L^p(\mathbb{R}^d) = \{[f] \mid \int |f|^p < +\infty\}$

§. Fubini's theorem.

Issue: $f(x,y)$ is a msb function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$

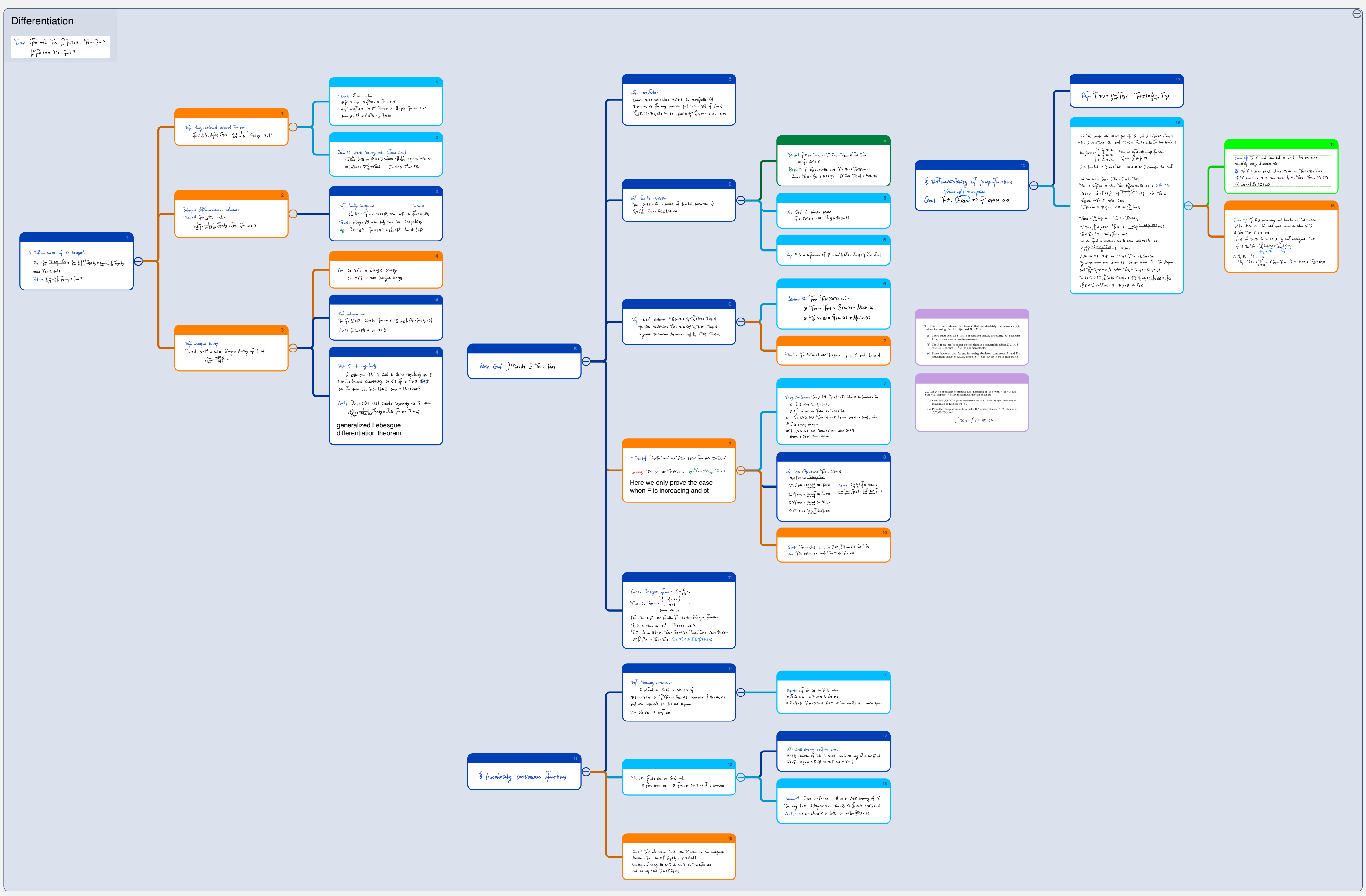
Goal: $\int f(x,y)\,dx\,dy$ "=" $\int dy \int f(x,y)\,dx$

Slice: $f^y(x) = f(x,y)$. y fixed

$E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$

Problem: E is squeezed to lower dimension.

E^y maybe not msb for some y



Differentiation
Issue: $f(x)$ msb. $F(t)=\int_0^t f(x)\,dx$, $F'(x)=f(x)$?
$\int_a^b F'(x)\,dx = F(b)-F(a)$?
generalized Lebesgue differentiation theorem
Here we only prove the case when F is increasing and ct
Goal: $F\uparrow$, F cts $\Rightarrow F'$ exists a.e.
20. This exercise deals with functions F that are absolutely continuous on $[a,b]$ and are increasing. Let $A=F(a)$ and $B=F(b)$.
(a) There exists such an F that is in addition strictly increasing, but such that $F'(x)=0$ on a set of positive measure.
(b) The F in (a) can be chosen so that there is a measurable subset $E\subset[A,B]$, $m(E)=0$, so that $F^{-1}(E)$ is not measurable.
(c) Prove, however, that for any increasing absolutely continuous F, and E a measurable subset of $[A,B]$, the set $F^{-1}(E)\cap\{F'(x)>0\}$ is measurable.
21. Let F be absolutely continuous and increasing on $[a,b]$ with $F(a)=A$ and $F(b)=B$. Suppose f is any measurable function on $[A,B]$.
(a) Show that $f(F(x))F'(x)$ is measurable on $[a,b]$. Note: $f(F(x))$ need not be measurable by Exercise 20 (b).
(b) Prove the change of variable formula: If f is integrable on $[A,B]$, then so is $f(F(x))F'(x)$, and
$\int_A^B f(y)\,dy = \int_a^b f(F(x))F'(x)\,dx.$

考试：作业题反复刷

① 积分理论、可测集 ② 微分理论（复杂） ③ Optional：抽象测度（不考）

Office Hour: Thursday 14:30–16:30. 611 college of science

Set theory review

§1.1. Countable set

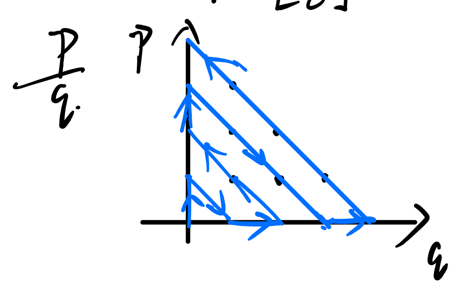
Def. Set S is called countable 当 $\exists$ bijective $S \to \mathbb{N}$.

Remark. 若 S is finite, we do not call it countable.

Countable + Finite := At most countable

Example. ① $\mathbb{Z}$ countable: $a \mapsto (-1)^a \lfloor \frac{a}{2} \rfloor$

② $\mathbb{Q}_{>0}$ countable $\frac{p}{q}$



③ $\mathbb{Q}$ countable.

$\mathbb{N} \sim \mathbb{Q}_{>0}$ $\quad$ $\mathbb{N} \sim \mathbb{Q}_{\le 0}$

$\mathbb{N} \xrightarrow{1:1} \mathbb{Z}_{>0} \cup \mathbb{Z}_{\le 0} \xrightarrow{1:1} \mathbb{Q}_{>0} \cup \mathbb{Q}_{\le 0} = \mathbb{Q}$

§1.2. uncountable set.

Claim $\mathbb{R}$ is uncountable.

Proof. (Diagonal method): WTS: $[0,1]$ is uncountable

Otherwise: $\exists\, \mathbb{N} \xrightarrow{1:1} [0,1]$ $\qquad a = 1.37500\ldots$

	0	1	2	3	4	
$f(1)$	0+1	0	2	3	4	$\ldots$
$f(2)$	0.	1+1	2	7	3	$\ldots$
$f(3)$	0.	3	7+1	5	9	$\ldots$

$d = 0.\overset{d_1}{1}\overset{d_2}{2}\overset{d_3}{8}\ldots$ $\qquad$ Claim: $d \neq f(k)$. $\forall k \in \mathbb{N}$

若 $d = f(k)$. Then $d_k = f(k)_k$

However, By def of d. $d_k = f(k)_k + 1$ or $0 \neq f(k)_k$

Contradiction! Therefore. $[0.1]$ is uncountable.

If $\exists\, \pi: \mathbb{N} \xrightarrow{1:1} \mathbb{R}$. then $S := \pi^{-1}[0.1]$ is an infinite set.

$\pi^{-1}[0.1] \subseteq \mathbb{N}$,

pick $a_1 = \min\{S\}$ $a_2 = \min\{S - a_1\}$ $\cdots\cdots$ we get a bijection

$\Rightarrow S$ is countable $\Rightarrow [0.1]$ is countable. Contradiction! $\square$

§ 1.3. Power set:

Def. S: set $P(S) := \{A \mid A \subseteq S\}$

Claim $P(S) \xrightarrow{1\text{-}1} \{f \mid f: S \to \{0,1\}\}$

$$A \subset S \mapsto f_A : a \mapsto \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases} \qquad A = \{f^{-1}(1)\} \to f.$$

$\#|P(S)| = 2^{\#|S|}$

Thm There is no bijection between S and $P(S)$.

Pf. If $\exists\, \theta: S \to P(S)$ bijective.

Consider $A = \{x \in S \mid x \notin \theta(x)\}$

Claim $A \neq \theta(t)$, $\forall t \in S$. Otherwise $A = \theta(t)$

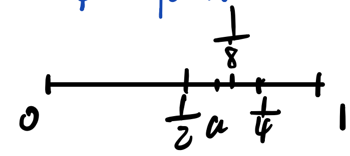
we consider t. ① $t \in A = \theta(t) \Rightarrow t \notin A$

② $t \notin A = \theta(t) \Rightarrow t \in A$

Remark. $\mathbb{R} \xrightarrow{1:1} P(\mathbb{N})$ By previous theorem $\mathbb{N} \not\xrightarrow{1:1} P(\mathbb{N}) \Rightarrow \mathbb{N} \not\xrightarrow{1:1} \mathbb{R}$

Claim. $[0.1] \xleftrightarrow{1:1} P(\mathbb{N})$

二进位展开



$$\overset{\Psi}{a} = 0.101001\cdots = \underbrace{\sum_{n=1}^{\infty} \frac{a_n}{2^n}} \quad a_n \in \{0,1\}$$

$a \longmapsto f_a: \mathbb{N} \to \{0,1\}$ $n \to a_n$

§. Measure Theory (测度)

$x \in \mathbb{R}^d$ $\quad |x| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$.

$E, F \subset \mathbb{R}^d$ $\quad d(E,F) = \inf\{|x-y| \mid x \in E, y \in F\}$

Def. open ball $B_r(x) := \{y \in \mathbb{R}^d \mid |y-x| < r\} \subseteq \mathbb{R}^d$

$E \subseteq \mathbb{R}^d$ is open iff $\forall x \in E$, $\exists r > 0$, $B_r(x) \subseteq E$

E is closed iff $E^c := \mathbb{R}^d / E$ open

E is compact $\iff$ closed + bounded

E bounded iff $\exists B_r(x) \supseteq E$

Remark E compact iff $\forall E \subseteq \bigcup_{j \in I} U_j$, $U_j \subseteq \mathbb{R}^d$ open

$\Rightarrow \exists$ finite set $J \subset I$. s.t. $E \subseteq \bigcup_{j \in J} U_j$

Open set in E. (induced topology)

$\{F \subset E \text{ open set}\} = \{F = E \cap U,\ U \subseteq \mathbb{R}^d \text{ open}\}$

Limit point: $x \in \mathbb{R}^d$ $\quad B_r(x) \cap E \neq \emptyset$. $\forall r \in \mathbb{R}_{>0}$

$x_i \in E$, $|x_j - x| < r$

Closure: $\bar{E} = \{x \in \mathbb{R}^d \mid B_r(x) \cap E \neq \emptyset, \forall r > 0\}$

Isolated point $x \in E$ & $\exists r > 0$. $B_r(x) \cap E = \{x\}$.

Interior point: $x \in E$ s.t. $\exists r > 0$. $B_r(x) \subseteq E$

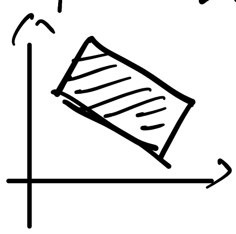
Relative interior point: let $E = [0,1]$

Interior point of (E in $\mathbb{R}^2$) $= \emptyset$

Interior point of (E in $\mathbb{R}^1$) $= E$

§. Rectangle & Cubes

Rectangle in $\mathbb{R}^d$: $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

Remark:  not a rectangle.

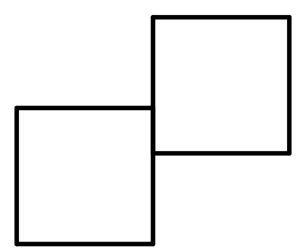
Volume: $|R| = \prod_{i=1}^{d} |b_i - a_i|$

Cube: a rectangle s.t. $|b_j - a_j| = c \quad \forall j$.

Almost disjoint $\{R_j \mid j \in J\}$. R_j: rectangle

They are almost disjoint 若 $R_i^\circ \cap R_j^\circ = \emptyset$.

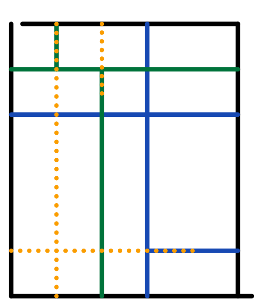
where R_j° is the set of interior point.



Lemma: If $R = \bigcup_{i \in I} R_i = \bigcup_{j \in J} R_j$

s.t. $\{R_i \mid i \in I\}$ almost disjoint $\{R_j \mid j \in J\}$ also ...

$\Rightarrow \sum_{i \in I} |R_i| = \sum_{j \in J} |R_j|$ we can define $|R| = \sum_{i \in I} |R_i|$

Pf.  (共取加细)

§. 2.3. Structure of open sets

$E \subseteq \mathbb{R}^d$. $\exists r>0$. $D_r(x) \subseteq E$

2.3.1. $U \subseteq \mathbb{R}^1$ open iff U can be uniquely written as at most countable disjoint union of open intervals

Pf. $x \in U$. $a_x = \inf\{a \mid (a,x) \subseteq U\}$ $b_x = \sup\{b \mid (a,b) \subseteq U\}$.

$x \in (a_x, b_x) \subseteq U$

Claim: either $(a_x, b_x) = (a_y, b_y)$ or $(a_x, b_x) \cap (a_y, b_y) = \phi$

existence: $U = \bigcup_x (a_x, b_x)$

uniqueness: $x \in U \in \bigcup (a_j', b_j')$ $x \in (a_j', b_j') \subseteq (a_x, b_x) \ni b_j' \in U$

Countable: $\exists q_x \in (a_x, b_x)$ $(a_x, b_x) \mapsto q_x \in \mathbb{Q}$

$\{(a_x, b_x)\} \xrightarrow[1:1]{\theta} S \subseteq \mathbb{Q}$

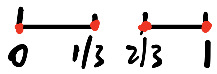
Thm $U \subseteq \mathbb{R}^d$ open set, U can be written as $U = \bigcup_{j \in I} R_j$. which is an almost disjoint union of closed cubes. I is countable

Recall. ① $U \subseteq \mathbb{R}^1$ open set

The Cantor Set

C_0: 

2^0 → end point.

C_1: 

2^1

- - - —

C_k $\quad$ 2^k intervals. length of interval $= \frac{1}{3^k}$

$C = \bigcap_{k=0}^{\infty} C_k$

Property.

1. $C \neq \phi$. end point of $C_k \in C$
2. C is closed: C_k closed for all k
3. C is totally disconnected: $\forall x, y \in C$ $[x, y] \not\subseteq C$

Pf Assume $C \supseteq [x, y] \supseteq [\frac{l}{3^k}, \frac{l+1}{3^{k+1}}] \subseteq C_k$

4. C does not have isolated pt.

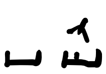
$\forall y \in C \Rightarrow y \in C_k$, $\exists$ end point $z \in C_k$. $|y - z| < \frac{1}{3^k}$, $\forall k$

5. C is uncountable

C is bijective to $[0, 1]$ and $P(\mathbb{N}) = \{f \mid f: \mathbb{N} \to [0, 1]\}$

Reason: $x \in C \Leftrightarrow x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ $\quad a_n \in \{0, 2\}$ 三进制展开

C_1:  $\quad x = \frac{0}{3} + \varepsilon_1$

C_2.  $\quad x = \frac{0}{3} + \frac{2}{3^2} + \varepsilon_2$

- - — — - $\quad x = \sum \frac{a_k}{3^k}$

$x_1 = \frac{a_1}{3}$ $a_1 \in \{0, 2\}$ Define $x_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$ $x_n \in C_n$.

$x_2 = \frac{a_1}{3} + \frac{a_2}{3^2}$ $x_n \to x \in C$

$\{x \in C\} \xleftrightarrow{1:1} \sum_{k=1}^{\infty} \frac{a_k}{3^k}, a_k \in \{0, 2\} \xleftrightarrow{1:1} \{f : \mathbb{N} \to \{0, 2\}\} \xleftrightarrow{1:1} P(\mathbb{N}) \xleftrightarrow{1:1} [0, 1]$

6. "Length" of C $C \subseteq C_k$ $m(C) \le m(C_k) = \frac{2^k}{3^k} \to 0$ as $k \to \infty$

§. 2.2. The exterior measure (outer measure) 外测度

$E \subseteq \mathbb{R}^d$. any set

Def Exterior measure. $m_*(E) = \inf\{\sum_{j=1}^{\infty} |Q_j| \mid E \subseteq \bigcup_{j=1}^{\infty} Q_j, Q_j\}$ cube

e.g. $m_*(\mathbb{R}^d) = \infty$. $\subseteq \mathbb{R}^2$ $m_*(E) \le \sum_{k \in \mathbb{Z}} 1 \cdot \frac{\varepsilon}{2^{|k|}} = 3\varepsilon$. $\varepsilon \to 0$

Remark. 1 $m_*(E)$ depends on E and $\mathbb{R}^d$

Consider $E = \mathbb{R}^1 \subseteq \mathbb{R}^2$. $m_*(E) \le \sum_{k \in \mathbb{Z}} 1 \cdot \frac{\varepsilon}{2^{|k|}} = \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \to 0$

Consider $E = \mathbb{R}^1 \subseteq \mathbb{R}^1$ $m_*(E) = \infty$ 不同的大空间取的cube不同

有限个 Q_j.

Remark 2. $m(E) = \inf\{\sum |Q_j| \mid E \subseteq \bigcup_{j \in \mathbb{N}} Q_j\} \ne \inf\{\sum |Q_j| \mid E \subseteq \bigcup_{j=1}^{N} Q_j\}$

① $E = \mathbb{Q} \cap (0, 1)$ $m_*(E) = 0$ 允许可数个覆盖.

$\mathbb{Q}$: countable $\Rightarrow$ E countable $E = \{a_1, a_2, \cdots a_j \cdots \mid j \in \mathbb{N}\}$

For any $a_i \in E$, take $Q_j \ni a_i$. $|Q_j| \le \frac{\varepsilon}{2^j}$

Then $m_*(E) = \sum_{j=1}^{n} |Q_j| \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon \to 0$

② $E \subseteq \bigcup_{j=1}^{N} Q_j$ (finite cover) Claim: $(0, 1) \subseteq \bigcup_{j=1}^{N} Q_j$

Otherwise: $(0, 1) - \bigcup_{j=1}^{N} Q_j = \underset{\text{open}}{(0, 1)} \cap \Big(\overset{\text{open}}{\bigcup_{j=1}^{N} Q_j}\Big)^c \ne \phi$.

Then $\exists\ (\alpha,\beta) \subseteq (0,1) - \bigcup_{j=1}^{N} Q_j \Rightarrow \exists\ q \in (\alpha,\beta)$ and $q \in \mathbb{Q}$

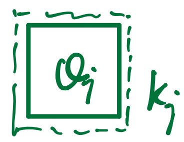
Contradiction! Therefore $(0,1) \subseteq \bigcup_{j=1}^{N} Q_j$

Example

① Q: cube $\Rightarrow m_*(Q) = |Q|$

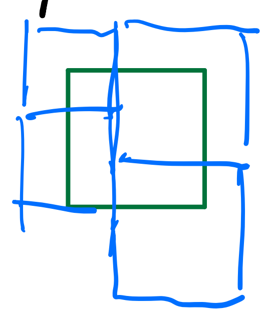
Since $Q \subseteq Q$ $\quad m_*(Q) = \inf\{\sum |Q_j| \mid Q \subseteq \bigcup_{j=1}^{\infty} Q_j\} \leq |Q|$

WTS: $\forall \varepsilon > 0 \quad m_*(Q) + \varepsilon \geq |Q|$



$\exists \bigcup_{j=1}^{\infty} Q_j \supseteq Q$ s.t. $\sum |Q_j| \leq m_*(Q) + \varepsilon$

Define $K_j \supseteq Q_j$, being a open cube slightly larger than Q_j.



$|K_j| - |Q_j| \leq \frac{\varepsilon}{2^j}$, By compactness $\exists N \quad \bigcup_{j=1}^{N} K_j \supseteq Q$

$\sum_{j=1}^{N} |K_j| \geq \sum_{j=1}^{N} \underbrace{|K_{js}'|}_{\text{refinement of } K_j} \geq |Q|$.

② R: rectangle: similar to cube

③ $m(\mathbb{R}^d) = \infty$

④ C: Cantor set $m_*(C) = 0$

Properties

1. Monotony: $E_1 \subset E_2 \Rightarrow m_*(E_1) \leq m_*(E_2)$

WTS: $\forall \varepsilon > 0 \quad m_*(E_2) + \varepsilon \geq m_*(E_1)$

$\exists \bigcup Q_j \supseteq E_2$. s.t. $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E_2) + \varepsilon$

Since $\bigcup Q_j \supseteq E_2 \supseteq E_1 \quad m_*(E_1) \leq \sum_{j=1}^{\infty} |Q_j| \leq m_*(E_2) + \varepsilon$

2. Countable additivity: $E = \bigcup_{j=1}^{\infty} E_j \Rightarrow m_*(E) \le \sum_{j=1}^{\infty} m_*(E_j)$

It suffice to show $\forall \varepsilon > 0$, $m_*(E) \le \sum_{j=1}^{\infty} m_*(E_j) + \varepsilon = \sum_{j=1}^{\infty}(E_j + \frac{\varepsilon}{2^j})$

$\exists \bigcup_k Q_k^{(j)} \supseteq E_j$ s.t. $\sum_{k=1}^{\infty} |Q_k^{(j)}| \le m_*(E_j) + \frac{\varepsilon}{2^j}$

Then $\bigcup_j \bigcup_k Q_k^{(j)} \supseteq \bigcup E_j = E \Rightarrow m_*(E) \le \sum_{j,k} |Q_k^{(j)}| = \sum_{j=1}^{\infty} m_*(E_j) + \varepsilon$

3. $E \subseteq \mathbb{R}^d$, $m_*(E) = \inf\{m_*(U) \mid U \supseteq E,\ U \text{ open}\}$

① $E \subseteq U$. $m_*(E) \le m_*(U)$. $m_*(E) \le \inf\{m_*(U)\}$

② WTS. $\forall \varepsilon > 0$. $\exists U \supseteq E$ $m_*(E) + \varepsilon \ge m_*(U)$

$\exists \bigcup Q_j \supseteq E$. s.t. $\sum |Q_j| \le m_*(E) + \varepsilon$ take K_j. $|K_j| - |Q_j| < \frac{\varepsilon}{2^j}$

then $U = \bigcup_j K_j \supseteq \bigcup_j Q_j \supseteq E$ $U \le \bigcup_j |K_j|$

$m_*(U) \le m_*(\bigcup_j K_j) \le \sum_j m_*(K_j) \le \sum_j (|Q_j| + \frac{\varepsilon}{2^j}) = \sum_j |Q_j| + \varepsilon \le m_*(E) + \varepsilon$

4. $E_1, E_2 \subseteq \mathbb{R}^d$ $0 < d(E_1, E_2) = \inf\{|x-y| \mid x \in E_1, y \in E_2\}$,

then $E = E_1 \cup E_2$, then $m_*(E) = m_*(E_1) + m_*(E_2)$

WTS: $m_*(E) + \varepsilon \ge m_*(E_1) + m_*(E_2)$. $\forall \varepsilon > 0$

$\exists \bigcup_j Q_j \supseteq E$, s.t. $\sum_j |Q_j| \le m_*(E) + \varepsilon$

Refine Q_j s.t. length of each $Q_j < \frac{\ell}{3^d}$ then

Then there is no j such that $Q_j \cap E_1 \ne \phi$ and $Q_j \cap E_2 \ne \phi$

$J_1 = \{Q_j \mid Q_j \cap E_1 \ne \phi\}$. $J_2 = \{Q_j \mid Q_j \cap E_2 \ne \phi\}$.

$m_*(E_1) \le \sum_{Q_j \in J_1} |Q_j|$ $m_*(E_2) \le \sum_{Q_j \in J_2} |Q_j|$

$m_*(E_1) + m_*(E_2) \le \sum_{Q_j \in J_1 \cup J_2} |Q_j| \le \sum |Q_j| \le m_*(E) + \varepsilon$

Exercise 1, 2, 3, 4

5. $E = \bigcup_{i=1}^{\infty} Q_i$ at most countable; almost disjoint cubes.

$\Rightarrow m_*(E) = \sum_{i=1}^{\infty} m_*(Q_i)$

Pf: "$\le$": sub-additivity.

"$\ge$": Find $P_i \subseteq Q_i^{\circ}$ s.t. P_i closed cube & $m_*(Q_i) \le m_*(P_i) + \frac{\varepsilon}{2^i}$

Then $P_1 \cdots P_n$ disjoint. $d(P_i, P_j) > 0$ $d(\bigcup_{i=1}^{N-1} P_i, P_N) > 0$

$m_*(\bigcup_{i=1}^{N} P_i) = \sum_{i=1}^{N} m_*(P_i)$

$m(\bigcup_{i=1}^{\infty} Q_i) \ge m(\bigcup_{i=1}^{N} Q_i) \ge m(\bigcup_{i=1}^{N} P_i) = \sum_{i=1}^{N} m(P_i) \ge \sum_{i=1}^{N} (m(Q_i) - \frac{\varepsilon}{2^i})$

$N \to \infty$: $m(\bigcup_{i=1}^{\infty} Q_i) \ge \sum_{i=1}^{\infty} m(Q_i) - \varepsilon$. $\varepsilon \to 0 \Rightarrow m(\bigcup_{i=1}^{\infty} Q_i) \ge \sum_{i=1}^{\infty} m(Q_i)$

§. Lebesgue Measurable set

Def. Lebesgue measurable set

$E \subseteq \mathbb{R}^d$. if $\forall \varepsilon > 0$ $\exists U \supseteq E$ open s.t. $m_*(U - E) < \varepsilon$

Then E is called Lebesgue measurable

Its Lebesgue measure is defined to be $m(E) = m_*(E)$

Property 1. open sets are measurable: $U = U$

Property 2. $m_*(E) = 0$ (measure zero set) $\Rightarrow E$ measurable

Property 3. Union of at most countably many measurable is still measurable

$E = \bigcup_{i=1}^{\infty} E_i$. $\forall\, U_i \supseteq E_i$ open, $m_*(U_i - E_i) < \frac{\varepsilon}{2^i}$

$U = \bigcup_{i=1}^{\infty} U_i$ open $\supseteq E$ $\quad m_*(U - E) \le m(\bigcup(U_i - E_i)) \le \sum m_*(U_i - E_i) \le \varepsilon$

Property 4. closed sets are measurable.

Lemma. K: compact. R: closed, $K \cap R = \emptyset \Rightarrow d(K, R) > 0$.

Otherwise $k_i \in K_i$. $r_i \in R$ $|k_i - r_i| \to 0$. $k_i \to \bar{k}$ $\quad$ k_i 为取它的收敛子列.

$|r_i - r_j| \le |r_i - k_i| + |k_i - k_j| + |k_j - r_j| \le 3\varepsilon \quad r_i \to \bar{r} \in R$

$\lim_{i \to \infty} |k_i - r_i| = |\bar{k} - \bar{r}| = 0 \Rightarrow \bar{k} = \bar{r}$. contradiction

Remark. If, $K \cdot R$ closed. $K \cap R = \emptyset$. then $d(K, R)$ may be 0

Pf. $E \cap B_r(0) = E_r$. (bounded, closed) $\quad E = \bigcup_{r \in \mathbb{N}} E_r$

$m_*(E_r) \le m_*(B_r) < \infty$ $\quad$ It suffice to show that compact sets are measurable

$\forall \varepsilon > 0.\ \exists U \supseteq E$. s.t. $m_*(U) \le m_*(E) + \varepsilon$ $\quad$ By compactness $m_*(E) < \infty$ $\Rightarrow m(U) - m(E) \le \varepsilon$

$U - E$: open $\Rightarrow U - E = \bigcup_{i=1}^{\infty} Q_i$ countable union of almost bijective cubes

$\bigcup_{i=1}^{N} Q_i$ closed, E closed $\Rightarrow (\bigcup_{i=1}^{N} Q_i) \cap E = \emptyset \Rightarrow d(\bigcup_{i=1}^{N} Q_i, E) > 0$

compact

$m_*(\bigcup_{i=1}^{N} Q_i \cup E) = \sum_{i=1}^{N} m_*(Q_i) + m_*(E)$

Since $U \supseteq \bigcup_{i=1}^{N} Q_i \cup E$ $\quad m_*(U) \ge m_*(\bigcup_{i=1}^{N} Q_i) + m_*(E) = \sum_{i=1}^{N} m_*(Q_i) + m_*(E)$

$m_*(U - E) \le m_*(\bigcup_{i=1}^{\infty} Q_i) \le m_*(U) - m_*(E) < \varepsilon$

Property 5. E measurable $\Rightarrow E^c = \mathbb{R}^d / E$ measurable

E measurable: $\forall \varepsilon = \frac{1}{n}$. $n \in \mathbb{N}$, $\exists U_n \supseteq E$ s.t. $m_*(U_n - E) < \frac{1}{n}$

U_n^c closed $\Rightarrow$ measurable. $S := \bigcup_{n=1}^{\infty} U_n^c$ measurable. $S \subset E^c$

Note that $(E^c - S) \subset (U_n - E)$ for all n.

$m_*(E^c - S) \le m_*(U_n - E) < \frac{1}{n}$ for all n.

$E^c - S$ is measurable, $E^c = (E^c - S) \cup (S)$ measurable.

Remark. E measurable depends on $\mathbb{R}^d$

Any set in $\mathbb{R}^1$ is zero-measure set in $\mathbb{R}^2$

Property 6. $\bigcap_{i=1}^{\infty} E_i$ measurable if E_i measurable

$(\bigcap_{i=1}^{\infty} E_i)^c = \bigcup (E_i^c)$ measurable $\Rightarrow \bigcap_{i=1}^{\infty} E_i^c$ measurable

Up shot $\mathcal{M} = \{E \mid E \text{ msb}\}$

$\mathcal{M}$ is closed under countable union/intersection/complement

Thm $\{E_i \mid i \in \mathbb{N}\}$ is a set of msb sets $E_i \cap E_j = \phi$ if $i \ne j$

$\Rightarrow m(\bigcup_{i \in \mathbb{N}} E_i) = \sum_{i \in \mathbb{N}} m(E_i)$ Countable additivity

Pf: "$\geq$": Reduction 1: we can assume E_i are bounded.

Reason $E_i^r := E_i \cap (B_r - B_{r-1})$ $\bigcup_{r \in \mathbb{N}} E_i^r = E_i$ B_r. $r \in \mathbb{N}$.

$\bigcup E_i = \bigcup_i \bigcup_{r \in \mathbb{N}} E_i^r$. E_i^r bdd $m(E) = m(\bigcup \bigcup E_i^r) = \sum_i \sum_r m(E_i^r) = \sum m(E_i)$

Reduction 2: Only need to show if $E_1 \cap E_2 = \phi$, $m(E_1 \cup E_2) = m(E_1) + m(E_2)$

(Use mathematical induction)

Pf. E_i^c measurable $\Rightarrow \Omega_i \supseteq E_i^c$ open $m(\Omega_i - E_i^c) < \frac{\varepsilon}{2^i}$ $\Omega_i^c \subseteq E_i$

$E_i - \Omega_i^c = \Omega_i - E_i^c$ $m(E_i - \Omega_i^c) < \varepsilon / 2^i$

$m(E_i) - m(\Omega_i^c) \le m(E_i - \Omega_i^c) < \frac{\varepsilon}{2^i} \Rightarrow \sum_{i=1}^{N} m(\Omega_i^c) \ge \sum_{i=1}^{N} m(E_i) - \varepsilon$

Ω_i^c: compact and disjoint $\Rightarrow m(\bigcup_i^N \Omega_i^c) = \sum_i^N m(\Omega_i^c)$

Since $\bigcup_{i=1}^{N} \Omega_i^c \subseteq E$. $m(\bigcup_i^\infty E_i) \ge m(\bigcup_i^N \Omega_i^c) = \sum_{i=1}^{N} m(\Omega_i^c) \ge \sum_{i=1}^{N} m(E_i) - \varepsilon$

Let $N \to \infty$. We obtain $m(\bigcup_{i=1}^\infty E_i) \ge \sum_{i=1}^{\infty} m(E_i)$

Notation: $E_k \subseteq E_{k+1}$, $\forall k \in \mathbb{N}$. $E := \bigcup_{k \in \mathbb{N}} E_k$, $E_k \uparrow E$

② $E_k \supseteq E_{k+1}$. $\forall k \in \mathbb{N}$ $E = \bigcap_{k \in \mathbb{N}} E_k$, $E_k \downarrow E$

Cor. E_k, $k \in \mathbb{N}$. msb

i) $E_k \uparrow E \Rightarrow \lim_{n \to \infty} m(E_k) = m(E)$

ii) $E_k \downarrow E$, if $m(E_1) < \infty \Rightarrow \lim_{k \to \infty} m(E_k) = m(E)$

Pf. $S_k = E_k - E_{k-1}$, $k = 0$, $E_{-1} = \phi$?

$E = \bigcup S_k$, $E_k = (E_k - E_{k-1}) \cup E_{k-1} \Rightarrow m(E) = \sum_{i \in \mathbb{N}} m(S_k)$

$m(E_k) = m(S_k) + m(E_{k-1})$

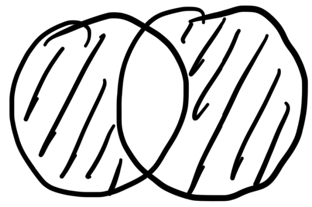
if $\exists j$. $m(E_j) = \infty$.

eg. $[i, +\infty)$ $E_i \downarrow \bigcap E_i = \phi$. $m(E) = 0$

eg. $E_i \downarrow E$. $\Rightarrow (E_1 - E_i) \uparrow (E_1 - E)$ $m(E_1 - E_i) \to m(E_1 - E)$

$m(E_1 - E_i + E_i) = m(E_1) < +\infty$. $m(E_1 - E_i) = m(E_1) - m(E_i)$

$m(E_i) \to m(E)$?



Notation Symmetric difference: Δ: $E \Delta F := (E-F)\cup(F-E)$

Thm E: measurable, $\varepsilon>0$.

① $\exists$ F closed, $F\subseteq E$, $m(E-F)<\varepsilon$

② If $m(E)<+\infty$, $\exists$ finite union of closed cubes $\bigcup_{i=1}^{N} Q_i$ s.t. $m(E\Delta(\bigcup_{i=1}^{N} Q_i))<\varepsilon$

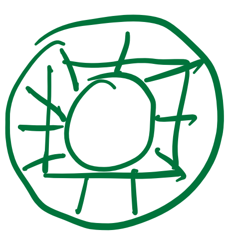
Pf ① E^c measurable. $\exists$ $\Omega\supseteq E^c$ open $m(\Omega-E^c)<\varepsilon$

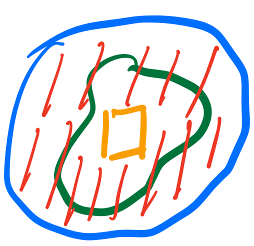
$\Omega-E^c=E-\Omega^c$ $\qquad m(E-\Omega^c)<\varepsilon$

①. $m(E)<\infty$. $E_r := E\cap B_r$. bdd set. $E_r\nearrow E$, $r\in\mathbb{N}$.

$m(E_r)\to m(E)<\infty$. $\exists N$. s.t. $r>N$. $m(E)-m(E_r)<\frac{\varepsilon}{2}$

By (i). $\exists$ $K\subseteq E_r$ closed. etc

Observation: 

② 

U-F: $\bigcup$ cubes

$E\subseteq\bigcup_{i=1}^{\infty} Q_i=\bigcup_{i=1}^{N} Q_i\bigcup_{i=N+1}^{\infty} Q_i$

$m(E\Delta\bigcup_{i=1}^{N} Q_i)\le m(E-\bigcup_{i=1}^{N} Q_i)+m(\bigcup_{i=1}^{N} Q_i-E)<2\varepsilon$

$m(E-\bigcup_{i=1}^{N} Q_i)\le m(\bigcup_{i=N+1}^{\infty} Q_i)<\varepsilon$, $\quad m(\bigcup_{i=1}^{N} Q_i-E)\le m(\bigcup_{i=1}^{\infty} Q_i-E)<\varepsilon$

Exercise: 14.15. Pb 4 (下周2完)

• $\mathbb{R}^d$: Vector space :

Property 1: Translation invariance: (平移)

$E \subseteq \mathbb{R}^d$ measurable; $\forall r \in \mathbb{R}^d$, $E + r := \{r + e \mid e \in E\} \subseteq \mathbb{R}^d$

$\Rightarrow$ $E + r$ measurable and $m(E+r) = m(E)$

Property 2: Dilation invariance.

$\delta \in \mathbb{R} > 0$. $\delta \cdot E := \{\delta \cdot e \mid e \in E\}$. msb and $m(\delta E) = \delta^d m(E)$

Property 3. $-E : \{-e \mid e \in E\}$. msb and $m(-E) = m(E)$

• σ-algebra: M is a collection of sets. which is closed under countable unions, complements and intersections.

Example. $M = \{E \subseteq \mathbb{R}^d \mid E \text{ msb}\}$.

• Borel σ-algebra:

The smallest σ-algebra containing open sets in $\mathbb{R}^d$

(= The σ-algebra generated by open sets in $\mathbb{R}^d$)

M_i: σ-algebra

$\bigcap_i M_i$: Intersection of all σ-algebra containing open sets.

$\{\bigcap_{i \in \mathbb{N}} U_i \mid U_i \in \mathbb{R}^d . \text{open}\} \to \{\bigcup_{\mathbb{N}} \bigcap_{\mathbb{N}} U_i \quad \}$?

Def. Borel set: $E \in$ Borel σ-algebra $\Leftrightarrow$ E is Borel set.

Cor: E measurable $\Rightarrow \exists F$. Borel set $m(E \Delta F) = 0$.

Def. G_δ-set: Countable intersection of open sets.

F_σ-set: Countable union of closed sets

Cor. $E \subseteq \mathbb{R}^d$ ① E is msb iff $\exists G_\delta$-set V s.t. $m(E \Delta V) = 0$

② E is msb iff $\exists F_\sigma$-set F s.t. $m(E \Delta F) = 0$

Pf. "$\Rightarrow$" $\forall \varepsilon = \frac{1}{n}$ $\exists U_n \supseteq E$. s.t. $m(U_n - E) < \frac{1}{n}$

Let $G = \bigcap_{n \in \mathbb{N}} U_n$ G_δ-set

"$\Leftarrow$": $U_n \supseteq G \supseteq E \Rightarrow m(G \Delta E) = m(G - E) \leq m(U_n - E) < \frac{1}{n}$

$E - G \subseteq E \Delta G \Rightarrow m(E - G) = 0,\ m(G - E) = 0.$

$E = (E - G) \cup (E \cap G) = \underbrace{(E - G)}_{\text{msb}} \cup (G - \underbrace{(G - E)}_{\text{msb}})$

- Construction of an unmeasurable set.

Equivalence class: $[a] := \{b \in S \mid a \sim b\}$.

Either $[a] = [b]$ or $[a] \cap [b] = \emptyset$

Define $\sim$ on $S = [0,1]$: $a \sim b$ iff $a - b \in \mathbb{Q}$

$[a] = \{a + r \in [0,1] \mid r \in \mathbb{Q}, a \in S\}$ $S = \bigcup [a]$

$N = \{a \mid a \text{ is the only element chosen in } [a]\}$

Claim: N is not msb.

$[0,1] \subseteq \bigcup_{r \in [-1,2] \cap \mathbb{Q}} (N+r) \subseteq [-1,3]$,

($\forall x \in [0,1]$. $x \in [a] \Rightarrow x = a + r$ $x - a = r$ $-1 \le r \le 1$)

$(N+r_1) \cap (N+r_2) = \phi$ if $r_1 \neq r_2$

Otherwise $x \in (N+r_1) \cap (N+r_2)$ $x = a_1 + r_1 = a_2 + r_2 \Rightarrow a_1 \sim a_2$, $a_1 = a_2$

If N is msb. then $m(\bigcup_{r \in [-1,2] \cap \mathbb{Q}} (N+r)) = \sum m(N) \le 4 \Rightarrow m(N) = 0$

Contradict to $\sum m(N) \ge 1$.

• Measurable Function :

Continuous : $\mathbb{R}^d \xrightarrow{F} \mathbb{R}$ $\forall U \in \mathbb{R}$ open, $F^{-1}(U)$ open

$E \subseteq \mathbb{R}^d$, $E \xrightarrow{F} \mathbb{R}^1 \cup \{+\infty\}$.

(if $E \xrightarrow{F} \mathbb{R}^1$, then F is called a Finite-valued Function)

Def. F is msb if (开集的原像是可测集)

① E is msb *不一定可测的原像是可测. (于E上)

② $\forall a \in \mathbb{R}$, $\{F < a\} := F^{-1}([-\infty, a))$ is msb

(if F is finite valued, $\{F < a\} := F^{-1}((-\infty, a))$, msb.

② $\Leftrightarrow$ ②', $\{F \le a\}$ is msb, $a \in \mathbb{R}$

Pf "$\Rightarrow$" $\{F \le a\} = \bigcap_{k \in \mathbb{N}} \{F < a + \frac{1}{k}\}$

"$\Leftarrow$": $\{F < a\} = \bigcup_{k \in \mathbb{N}} \{F \le a + \frac{1}{k}\}$

① <=> ③: $\{f>a\}$ msb: $\{x \mid f(x)>a\}^c = \{f>a\}^c = \{f\le a\}$ msb.

① <=> ③': $\{f\ge a\}$ msb.

<=> if f is finite-valued, then $f^{-1}((a,b))$ msb $a,b\in\mathbb{R}$

$\{f<a\} = \bigcup_{k\in\mathbb{N}}\{-k<f<a\}$. $\{a<f<b\} = \{f<b\}\cap\{f>a\}$

∘ f finite-valued ⇒ $f^{-1}(U)$ msb. $U\subseteq\mathbb{R}^1$ open

$f^{-1}(U) = \bigcup f^{-1}((a_i,b_i))$

∘ if E 到 $\mathbb{R}^1\cup\{\pm\infty\}$; f msb iff {开集下原像可测, ∞下原像可测。

① E msb ② $f^{-1}(U)$ msb, $\forall U\subseteq\mathbb{R}^1$ ③ $f^{-1}(-\infty)$ msb

Pf. "⇒" $f^{-1}(-\infty) = \{f<a\} - \bigcup_{k\in\mathbb{N}}\{-k<f<a\}$.

"⇐". $\{f<a\} = f^{-1}(-\infty)\cup\left(\bigcup_{k\in\mathbb{N}}\{-k<f<a\}\right)$

① Characteristic function: $E\subseteq\mathbb{R}^d$, $\chi_E(x) = \begin{cases}1, & x\in E\\ 0, & x\notin E\end{cases}$

Simple function: $E_i\subseteq\mathbb{R}^d$ msb sets $i=1\cdots k$. $f = \sum_{i=1}^{k} a_i\chi_{E_i}$, $a_i\in\mathbb{R}$.

Step function. $R_i\subseteq\mathbb{R}^d$ rectangle, $i=1\cdots k$. $f = \sum_{i=1}^{k} a_i\chi_{R_i}$, $a_i\in\mathbb{R}$

Property 1 Cts functions $f: \mathbb{R}^d\to\mathbb{R}$ (finite) are measurable

Property 2 $\Phi\circ f$ is msb. Φ: Cts. f: msb. finite

Warning: $f\circ\Phi$ may not be msb

Ex. 10, 11, 25, 26, 21, 32, 33
Prob. 5*.

Prop 3. $\{f_n\}_{n\in\mathbb{N}}$ msb, then:

$\sup_n f_n(x)$, $\inf_n f_n(x)$, $\limsup_{n\to\infty} f_n(x)$, $\liminf_{n\to\infty} f_n(x)$ are msb

Pf. $\{\sup f_n(x) > a\} = \bigcup_n \{f_n > a\}$ msb.

$\inf_n f_n(x) = -\sup_n(-f_n(x))$

$\limsup_{n\to\infty} f_n = \inf_k \{\sup_{n\geq k} f_n\}$ $\liminf_{n\to\infty} f_n = \sup_k \{\inf_{n\geq k} f_n\}$

Prop 4. $f_n \to f$, f_n msb $\Rightarrow$ f msb.

Pf. $f(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$ msb

Prop 5. msb functions are closed under arithmetic.

If f and g are msb, then: ① $\forall n\in\mathbb{N}$. f^n, ② $f\pm g$. ③ $f\cdot g$ are msb

① $\{f^n > c\} = \begin{cases} \{f > c^{\frac{1}{n}}\}, & n \text{ odd} \\ \{|f| > c^{\frac{1}{n}}\} = \{f > c^{\frac{1}{n}}\} \cup \{f < -c^{\frac{1}{n}}\} & n \text{ even} \end{cases}$

② $\{f+g > c\} = \bigcup_{r\in\mathbb{Q}} \{f > a-r\} \cap \{g > r\}$. ($\mathbb{Q}$ is countable).

③ $f\cdot g = \dfrac{(f+g)^2 - (f-g)^2}{4}$ msb.

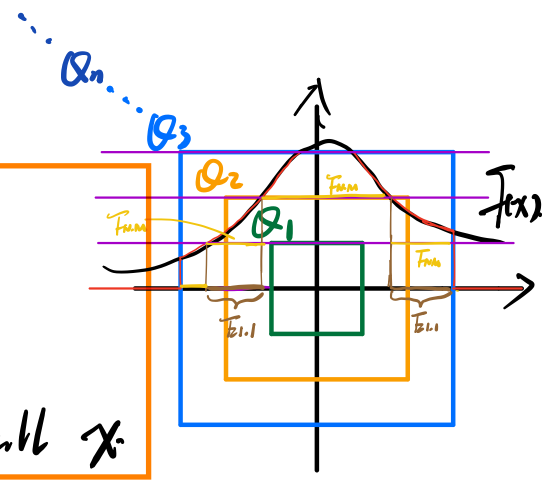
Def Almost everywhere equal (An equivalence relation)

$f = g$ a.e. $x\in E$ $\Leftrightarrow$ $m\{f\neq g\} = 0$

Prop 6. $f = g$ a.e. $x \in E$, then f msb $\Rightarrow$ g msb.

Pf. $\{g < c\} \,\Delta\, \{f < c\} \subseteq \{f \neq g\}$ $\quad m(\{f \neq g\}) = 0$

§. Approximation by Simple Functions.



Thm 4.1 $f \geq 0$, msb (f can be $+\infty$), then:

$\exists \{\varphi_k\}$, $k \in \mathbb{N}$, simple functions ($\sum a_i \chi_{E_i}$) s.t.

$\varphi_k(x) \leq \varphi_{k+1}(x)$ and $\lim_{k\to\infty} \varphi_k(x) = f(x)$ for all x.

Pf. Define $F_N(x) = \begin{cases} f(x), & \text{if } x \in Q_N \text{ and } f(x) \leq N \\ N & \text{if } x \in Q_N \text{ and } f(x) > N \\ 0, & \text{otherwise} \end{cases}$

Then $\lim_{N\to\infty} F_N(x) \to f(x)$. For fixed $M, N \geq 1$ we define

$E_{\ell,M} = \{x \in Q_N : \frac{\ell}{M} < F_N(x) \leq \frac{\ell+1}{M}\}$ for $0 \leq \ell \leq NM$

Then we can write $F_{N,M}(x) = \sum_{\ell} \frac{\ell}{M} \chi_{E_{\ell,M}}(x)$

Each $F_{N,M}$ is a simple function s.t. $0 \leq F_N(x) - F_{N,M}(x) \leq \frac{1}{M}$, $\forall x$

If we chose $N = M = 2^k$ with $k \in \mathbb{N}^+$, let $\varphi_k = F_{2^k, 2^k}$, then

$0 \leq F_M(x) - \varphi_k(x) \leq \frac{1}{2^k}$ for all x, and $\{\varphi_k\}$ is increasing

Thm 4.2 f msb on $\mathbb{R}^d$, then $\exists \{\varphi_k\}_{k=1}^{\infty}$ s.t.

$|\varphi_k(x)| \le |\varphi_{k+1}(x)|$ and $\lim_{k\to\infty} \varphi_k(x) = f(x)$ for all x.

Prf: pre $f = f^+(x) - f^-(x)$..

where $f^+(x) = \max(f(x), 0) \ge 0$ $\quad f^-(x) = \max(-f(x), 0) \ge 0$

By Thm 4.1, there exists $\{\varphi_k^{(1)}(x)\}_{k=1}^{\infty}$, $\{\varphi_k^{(2)}(x)\}_{k=1}^{\infty}$ converging to f^+, f^-

Define $\varphi_k = \varphi_k^{(1)} - \varphi_k^{(2)}$, then $\varphi_k \to f$

Since $|\varphi_k(x)| = \varphi_k^{(1)}(x) + \varphi_k^{(2)}(x)$. $|\varphi_k(x)|$ is increasing.

Thm 4.3 f msb on $\mathbb{R}^d$. Then $\exists$ step functions $\psi_k \to f$ a.e.

Prf. Since f can be approximated by simple functions.

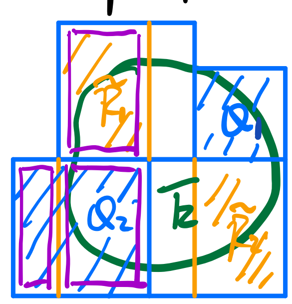
It suffice to show χ_E can be approximated by step functions.

For msb set E.

$\forall \varepsilon > 0$, $\exists$ cubes $Q_1 \cdots Q_N$ s.t. $m(E \triangle \bigcup_j Q_j) < \varepsilon$

By extending the sides of the cubes, we obtain

almost disjoint rectangles $\tilde{R}_1 \cdots \tilde{R}_M$ s.t. $\bigcup_j^N Q_j = \bigcup_j^M \tilde{R}_j$



Taking rectangles R_j contained in $\tilde{R}_j$ and slightly smaller,

we obtain a collection of disjoint rectangles s.t. $m(E \triangle \bigcup_{j=1}^M R_j) < 2\varepsilon$.

Therefore $\chi_E = \sum_{j=1}^M \chi_{R_j}(x)$, except possibly on a set of measure $< 2\varepsilon$

Consequently, $\forall k \in \mathbb{N}^+$, $\exists \psi_k$ (step) s.t. $m(E_k) \le 2^{-k}$

$E_k := \{x : \chi_E(x) \ne \psi_k(x)\}$

Let $F_k = \bigcup_{j=k+1}^{\infty} E_j$ $F = \bigcap_{k=1}^{\infty} F_k$, then $m(F_k) \le 2^{-k}$, $m(F) = 0$

$\varphi_k \to \chi_E(x)$ for all $x \in F^c$. Q.E.D.

§. Littlewood's three principles:

1. Every msb set is "nearly" a finite union of intervals.
2. Every msb function is "nearly" a cts function
3. Every convergent sequence is "nearly" uniformly convergent.

Thm 4.4 Egorov theorem

$\{f_k\}_{k\ge1}$ msb on E with $m(E) < \infty$, $f_k \to f$ a.e. on E, then:

$\forall \varepsilon > 0$. $\exists$ closed $A_\varepsilon \subset E$ s.t. $m(E - A_\varepsilon) < \varepsilon$ and $f_k \xrightarrow{u} f$ on A_ε

Pf: W.L.O.G., we assume $f_k \to f$ for every $x \in E$

Define $E_k^n = \{x \in E : |f_j(x) - f(x)| < \frac{1}{n}$ for all $j > k\}$

Fix n, then $E_k^n \subset E_{k+1}^n$, $E_k^n \uparrow E$ as $k \to \infty$

$\exists k_n$ s.t. $m(E - E_{k_n}^n) < \frac{\varepsilon}{2^n}$, then we have

$|f_j(x) - f(x)| < \frac{1}{n}$ when $j > k_n$ and $x \in E_{k_n}^n$

Reason: $E - \tilde{A}_\varepsilon = E \cap \tilde{A}_\varepsilon^c$
$= E \cap (\bigcap E_{k_n}^n)^c = E \cap (\bigcup E_{k_n}^{n\,c})$
$= \bigcup (E \cap E_{k_n}^{n\,c}) = \bigcup (E - E_{k_n}^n)$

Let $\tilde{A}_\varepsilon = \bigcap_{n \in \mathbb{N}} E_{k_n}^n$. $m(E - \tilde{A}_\varepsilon) \le \sum_{n \in \mathbb{N}} m(E - E_{k_n}^n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

Claim: $f_k|_{\tilde{A}_\varepsilon} \xrightarrow{u} f|_{\tilde{A}_\varepsilon}$

$\forall \delta > 0$. $\exists \frac{1}{n} < \delta$. $\tilde{A}_\varepsilon \subseteq E^n_{k_n}$ when $j > k_n$ $|f_j(x) - f(x)| < \frac{1}{n} < \delta$

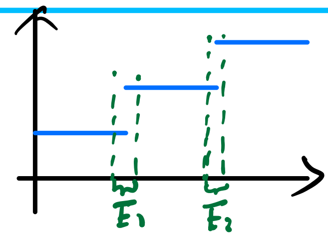
By Thm 3.4, we can find $A_\varepsilon \subset \tilde{A}_\varepsilon$, closed with $m(A_\varepsilon - \tilde{A}_\varepsilon) < \varepsilon$

Thm 4.5. (Lusin)

① f msb on E ② $|f| < \infty$ ③ $m(E) < \infty$ Then:

$\forall \varepsilon > 0$, $\exists$ closed $F_\varepsilon \subset E$ and $m(E - F_\varepsilon) < \varepsilon$ s.t. $f|_{F_\varepsilon}$ is cts

Warning: f and $f|_{F_\varepsilon}$ are different functions. F_ε may not be cts set of f



Pf. Let $\{f_n\}$ be step functions s.t. $f_n \to f$ a.e.

Then we can find sets E_n s.t. $m(E_n) < \frac{\varepsilon/3}{2^n}$

and f_n is cts outside E_n.

By Egorov's thm, we can find $A_{\varepsilon/3}$ on which $f_n \xrightarrow{u} f$, $m(E - A_{\varepsilon/3}) < \frac{\varepsilon}{3}$

Then we put $F' = A_{\varepsilon/3} - \bigcup_n E_n$.

Every f_n is cts on F'. by uniform convergence, f is also cts on F'

Then find a closed set $F_\varepsilon \subset F'$ s.t. $m(F_\varepsilon - F') < \frac{\varepsilon}{3}$.

Then $m(E - F_\varepsilon) \leq m(E - A_{\varepsilon/3}) + m(A_{\varepsilon/3} - F') + m(F' - F_\varepsilon) < \varepsilon$